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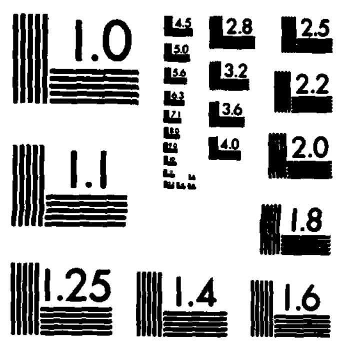


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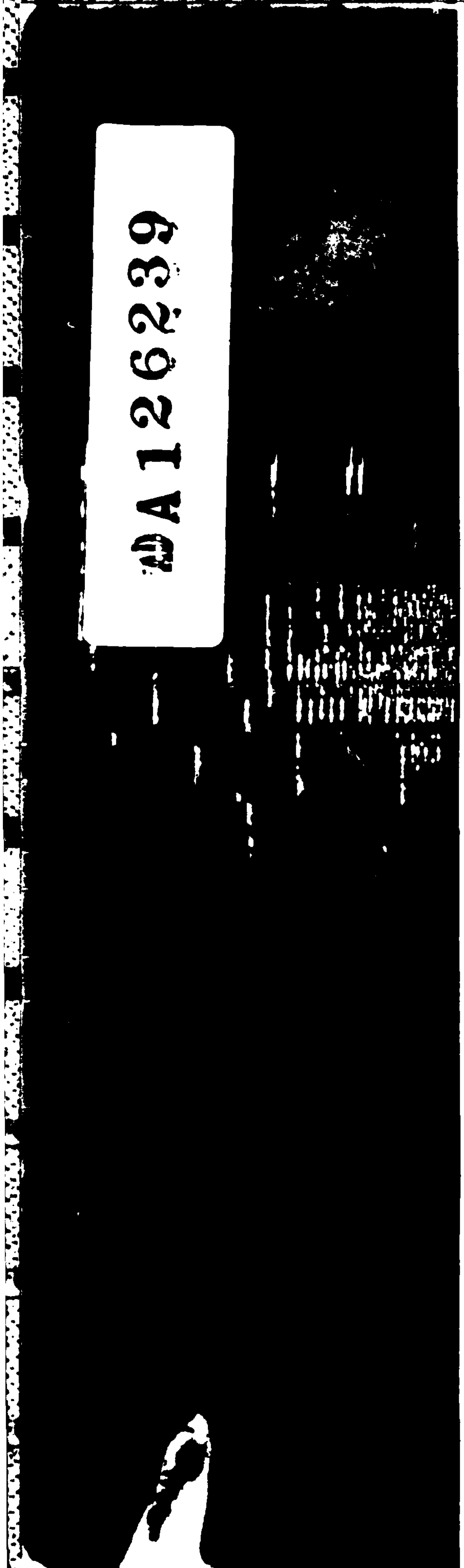



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by

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## Abstract

A new multiplicative efficiency formulation is developed wherein the efficiency values are invariant under changes in the units of measurement of outputs and inputs. It is shown that the associated Data Envelopment Analysis (DEA) implies that optimal envelopments are of piecewise Cobb-Douglas type. This leads to a new method for estimating frontier production functions of Cobb-Douglas type.

KEY WORDS

Multiplicative Efficiency
Log Efficiency
Data Envelopment Analysis
Piecewise Cobb-Douglas Forms
Production Functions

# Invariant Multiplicative Efficiency and Piecewise Cobb-Douglas Envelopments

by

A. Charnes, W. W. Cooper, L. Seiford, J. Stutz*

## Introduction

In [1], Charnes, Cooper, Seiford, and Stutz ($C^2S^2$) develop a multiplicative (orlog) measure of the relative efficiency of multiple input, multiple output productive (or "decisionmaking") units (DMU's). In contrast to the CCR measure [2, 3], the multiplicative measure obtained in [1] is not invariant under change of units in the inputs or outputs. We show here how by a simple change preserving the multiplicative format that a units invariant multiplicative measure can be obtained. Interestingly, the Data Envelopment Analysis (DEA) associated with this new modification necessarily yields optimal envelopments by Cobb-Douglas functions, i.e., the efficiency surface is piecewise Cobb-Douglas rather than merely log-linear! This uncovers a new role for Cobb-Douglas functions[1]--they are necessary for the units invariant property of a multiplicative measure.

## Units Invariant Multiplicative Efficiencies

The $C^2S^2$ multiplicative model reduces the input-output quantities to single virtual output to input ratios. If we now introduce an additional virtual output multiplier and virtual input multiplier, we obtain the following form for our problem to measure the efficiency of $DMU_o$ relative to all the n DMU's:

$$\max \left(e^{\hat{\eta}} \prod_{n=1}^{s} Y_{ro}^{\mu_r}\right) / \left(e^{\hat{\xi}} \prod_{i=1}^{m} X_{io}^{\nu_i}\right)$$

$$\text{(1)} \qquad \text{s.t.} \quad \left(e^{\hat{\eta}} \prod_{r=1}^{s} Y_{rj}^{\mu_r}\right) / \left(e^{\hat{\xi}} \prod_{i=1}^{m} X_{ij}^{\nu_i}\right) \leq 1, \; j = 1,\ldots,n$$

$$-\hat{\eta} \leq 0, \; -\hat{\xi} \leq 0, \; -\mu_r \leq -\delta, \; -\nu_i \leq -\delta, \; \forall r,i,$$

---

[1] Other properties relating Cobb-Douglas forms to more general classes of functions are examined in [5].

where $\delta > 0$ and $DMU_0$ is one of the n DMU's in the constraints.

Suppose the units in the outputs and the inputs are changed so that $Y_{rj}$ becomes $a_r Y_{rj}$ and $X_{ij}$ becomes $b_i X_{ij}$ where $a_r, b_i > 0$, $\forall r,i$ (Note $a_r$ or $b_i = 1$ corresponds to no change in those units). Problem (1) becomes

$$\max \quad e^{\hat{\eta}}(\prod_r a_r^{\mu_r})\prod_r Y_{ro}^{\mu_r} \Big/ e^{\hat{\xi}}(\prod_i b_i^{\nu_i})\prod_i X_{io}^{\nu_i}$$

$$(2) \qquad \text{s.t.} \quad e^{\hat{\eta}}(\prod_r a_r^{\mu_r})\prod_r Y_{rj}^{\mu_r} \Big/ e^{\hat{\xi}}(\prod_i b_i^{\nu_i})\prod_i X_{ij}^{\nu_i} \leq 1, \; j = 1,\ldots,n$$

$$-\hat{\eta} \leq 0, \; -\hat{\xi} \leq 0, \; -\mu_r \leq -\delta, \; -\nu_i \leq -\delta, \; \forall r,i.$$

If (1) has optimal value E(1) with $\eta^*$, $\xi^*$, $\mu^*$, $\nu^*$ an optimal solution, then $\exp(\hat{\eta}) = K\exp(\hat{\eta}^*)/\prod_r a_r^{\mu_r^*}$, $\exp(\hat{\xi}) = K\exp(\hat{\xi}^*)/\prod_i b_i^{\nu_i^*}$, $\mu^*$, $\nu^*$ (where $K > 0$ assures $\hat{\eta},\hat{\mu} > 0$) is "feasible" for (2) with value E(1). Hence for (2) the optimal value $E(2) \geq E(1)$. Similarly from an optimal solution $\bar{\eta}$, $\bar{\xi}$, $\bar{\mu}$, $\bar{\nu}$ to (2) we construct a feasible solution to (1) with value E(2). Thereby $E(1) \leq E(2) \leq E(1)$ i.e., the efficiency value is invariant under change of units.

## The Cobb-Douglas Property

Taking logarithms in (1) and going to vector matrix notation as in [1], we obtain the dual linear programming problems:

$$(3) \qquad
\begin{array}{llll}
 & \underline{\text{I}} & & \underline{\text{II}} \\
\max & \hat{\eta} - \hat{\xi} + \mu^T\hat{Y}_0 - \nu^T\hat{X}_0 & \min & -\delta e^T s^+ - \delta e^T s^- \\
\text{s.t.} & \hat{\eta}e^T - \hat{\xi}e^T + \mu^T\hat{Y} - \nu^T\hat{X} \leq 0 & \text{s.t.} & e^T\lambda - \theta^+ = 1 \\
 & -\hat{\eta} \leq 0 & & -e^T\lambda - \theta^- = -1 \\
 & -\hat{\xi} \leq 0 & & \hat{Y}\lambda - s^+ = \hat{Y}_0 \\
 & -\mu^T \leq -\delta e^T & & -\hat{X}\lambda - s^- = -\hat{X}_0 \\
 & -\nu^T \leq -\delta e^T & & \lambda, \theta^+, \theta^-, s^+, s^- \geq 0
\end{array}$$

Here II represents the DEA side of the efficiency problem. Adding the first two equations in II, we obtain $-\theta^+ - \theta^- = 0$. Since $\theta^+, \theta^- \geq 0$ we must have $\theta^+ = \theta^- = 0$. Thus II reduces to

$$
(4)\qquad
\begin{array}{ll}
\min & -\delta e^T s^+ - \delta e^T s^- \\
\text{s.t.} & \hat{Y}\lambda - s^+ = \hat{Y}_0 \\
 & -\hat{X}\lambda \quad -s^- = -\hat{X}_0 \\
 & e^T\lambda = 1 \\
 & \lambda, s^+, s^- \geq 0
\end{array}
$$

Thereby we have $\hat{Y}_0$ and $\hat{X}_0$ enveloped by convex combinations of the $\hat{Y}_j$, $\hat{X}_j$. With optimal solutions $\lambda^*$, $s^{*+}$, $s^{*-}$, we can write

$$
(5)\qquad
\begin{array}{l}
Y_0 = \prod_{j=1}^{n} Y_j^{\lambda_j^*} e^{-s_j^{*+}} \\
X_0 = \prod_{j=1}^{n} X_j^{\lambda_j^*} e^{s_j^{*-}}
\end{array}
\qquad \text{where } \sum_j \lambda_j^* = 1
$$

and by $Y_j^{\lambda_j^*}$, resp. $X_j^{\lambda_j^*}$, we mean $(Y_{1j}^{\lambda_j^*}, \ldots, Y_{sj}^{\lambda_j^*})^T$ resp. $(X_{1j}^{\lambda_j^*}, \ldots, X_{mj}^{\lambda_j^*})^T$. Thus our optimal envelopments are by Cobb-Douglas functions with $\lambda_j^* > 0$ implying that $DMU_j$ is efficient, i.e., $DMU_0$ is associated with the efficiency surface "facet" spanned by those $DMU_j$'s for which $\lambda_j^* > 0$.

We note further that the simplified dual programs corresponding to (3) are now

$$
(6)\qquad
\begin{array}{llll}
 & \underline{(I')} & & \underline{(II')} \\
\max & \mu^T\hat{Y}_0 - \nu^T\hat{X}_0 + \omega & \min & -\delta e^T s^+ - \delta e^T s^- \\
\text{s.t.} & \mu^T\hat{Y} - \nu^T\hat{X} + \omega e^T \leq 0 & \text{s.t.} & \hat{Y}\lambda - s^+ = \hat{Y}_0 \\
 & -\mu^T \leq -\delta e^T & & -\hat{X}\lambda - s^- = -\hat{X}_0 \\
 & -\nu^T \leq -\delta e^T & & e^T\lambda = 1 \\
 & & & \lambda, s^+, s^- \geq 0
\end{array}
$$

These results present us with a new method for estimating piecewise Cobb-Douglas production functions directly from empirical data. The form of (II') in contrast to that of [4] is also sufficiently simple that one can anticipate that the mathematical statistics of this type of Cobb-Douglas estimation may well be developed in the near future (see also the Appendix in [3]).

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